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SINGULAR LIMIT OF SOLUTIONS OF THE POROUS MEDIUM EQUATION WITH ABSORPTION

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Abstract. We prove that as $m \to \infty$ the solutions $u^{(m)}$ of $u_t = \Delta u^m - u^p$, $(x,t) \in R^n \times (0,T)$, $T > 0$, $m > 1$, $p > 1$, $u \geq 0$, $u(x,0) = f(x) \in L^1(R^n) \cap L^\infty(R^n)$, converges in $L^1_{loc}(R^n \times (0,T))$ to the solution of the ODE $v_t = -v^p$, $v(x,0) = g(x)$, where $g \in L^1(R^n)$, $0 \leq g \leq 1$, satisfies $g - \Delta \widetilde{g} = f$ in $\mathcal{D}'(R^n)$ for some function $\widetilde{g} \in L^\infty_{loc}(R^n)$, $\widetilde{g} \geq 0$, satisfying $\widetilde{g}(x) = 0$ whenever $g(x) < 1$ for a.e. $x \in R^n$, $\int_E \widetilde{g}dx \leq C|E|^{2/n}$ for $n \geq 3$ and $\int_E |\nabla \widetilde{g}|dx \leq C|E|^{1/2}$ for $n = 2$, where $C > 0$ is a constant and E is any measurable subset of R^n.

In this paper we will show that as $m \to \infty$ the solutions $u = u^{(m)}$ of the equation

$$\begin{cases} u_t = \Delta u^m - \lambda u^p, u \geq 0, & \text{in } R^n \times (0,T), \\ u(x,0) = f(x) \geq 0, x \in R^n, & f \in L^1(R^n) \cap L^\infty(R^n), \end{cases} \tag{0.1}$$

where $T > 0$, $p > 1$, $\lambda > 0$ converges to the solution of the following ODE:

$$\begin{cases} v_t = -\lambda v^p & \text{in } \mathcal{D}'(R^n \times (0,T)), \\ v(x,0) = g(x), & x \in R^n, \end{cases} \tag{0.2}$$

where $g \in L^1(R^n)$, $0 \leq g \leq 1$, satisfies

$$g - \Delta \widetilde{g} = f \quad \text{in } \mathcal{D}'(R^n) \tag{0.3}$$

for some function $\widetilde{g} \in L^\infty_{loc}(R^n)$, $\widetilde{g} \geq 0$, satisfying

$$\widetilde{g}(x) = 0 \quad \text{whenever } g(x) < 1 \text{ for a.e. } x \in R^n \tag{0.4}$$

and

$$\begin{cases} \displaystyle\int_E \widetilde{g}(x)dx \leq C|E|^{2/n} & \text{for } n \geq 3, \\ \displaystyle\int_E |\nabla \widetilde{g}(x)|dx \leq C|E|^{1/2} & \text{for } n = 2 \end{cases} \tag{0.5}$$

for any measurable set $E \subset R^n$, where $C > 0$ is a constant independent of E and $m > p$.

When $\lambda > 0$, equation (0.1) models the flow of gases through a porous medium or thermal propagation with absorption where both the diffusion coefficient and the absorption coefficient are powers of the concentration u [K1][K2][VW]. When

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$\lambda = 0$, the above equation reduces to the well-known porous medium equation [A][P].

For $\lambda = 0$, Caffarelli and A.Friedman [CF] studied the asymptotic behaviour of solutions of (0.1) and showed that the solutions of (0.1) converge as $m \to \infty$ if f satisfies (0.1) and the following conditions:

$$\begin{aligned} &f \in C^1 \text{ in supp } f, \\ &f(0) > 1, f_r < 0 \text{ in } R^n \setminus \{0\} \cap \text{supp } f, \\ &f_{r_{x_0}} \le 0 \text{ in } R^n \setminus B_1(0) \cap \text{supp } f \qquad \forall x_0 \in B_{\varepsilon_0}(0) \end{aligned}$$

for some $\varepsilon_0 > 0$, where $r_{x_0} = |x - x_0|$, $B_r(0) = \{x : |x| < r\}$ and $f_{r_{x_0}}$ is the radial derivative of f with center at x_0.

Their result was later generalized to the case of general $f \in L^1(R^n)$ by P.Bénilan, L.Boccardo and M.Herrero [BBH] and P.E.Sacks [S2]. Recently X.Xu [X] proved that one has similar result in the case of a hyperbolic equation. K.M.Hui [H1]–[H3] also obtained similar results in the case of the generalized p-Laplacian equation, the porous medium equation with convection term and in the case of equation (0.1) as $p \to \infty$.

Our result shows that the diffusion term is negligible compared with the other terms of the equation as $m \to \infty$. This is in sharp contrast to the case $p \to \infty$ [H1] in which the solution of (0.1) will converge to the solution of the porous medium equation with no absorption term.

We will organize our paper as follows. In section 1 we recall some existence and regularity results for equation (0.1) from [KPV] and [BC]. We will also recall a uniqueness result for (0.3) from [BBC] and prove some technical lemmas. In section 2 we will first prove the convergence result for the case $f \in C_0(R^n)$. We then prove our main convergence theorem by an approximation argument.

For simplicity we will assume that $T = 1$ and $\lambda = 1$ throughout the rest of the paper.

We start with some definitions. For any open set $\Omega_0 \subset R^n$, $h \in C(R)$, we say that u is a solution (respectively subsolution) of (cf. [DK])

$$u_t = \Delta u^m + h(u) \tag{0.6}$$

in $\overline{\Omega}_0 \times (0,1)$ if (a) u is continuous and non-negative in $\overline{\Omega}_0 \times (0,1)$, (b) $u \in L^\infty([0,1); L^1(\Omega_0)) \cap L^\infty(\Omega_0 \times (0,1))$ and (c) u satisfies

$$\int_{\tau_1}^{\tau_2}\int_\Omega \left[u^m \Delta\eta + u\frac{\partial \eta}{\partial t} + h(u)\eta\right] dxdt = \int_{\tau_1}^{\tau_2}\int_{\partial\Omega} u^m \frac{\partial \eta}{\partial n} d\sigma ds + \int_\Omega u\eta dx\bigg|_{\tau_1}^{\tau_2} \tag{0.7}$$

(respectively $\ge$) for all bounded open set $\Omega \subset \Omega_0$ with $\partial\Omega \in C^2$, $0 < \tau_1 \le \tau_2 < 1$, $\eta \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\eta \equiv 0$ on $\partial\Omega \times [\tau_1, \tau_2]$ where $\partial/\partial n$ is the exterior normal derivative on $\partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$.

If u is a solution of (0.6) in $\overline{\Omega}_0 \times (0,1)$, we say that u has initial trace or initial value $d\mu$ if

$$\lim_{t\to 0}\int u(x,t)\eta(x)dx = \int \eta d\mu \quad \forall \eta \in C_0^\infty(\overline{\Omega}_0)$$

We let $\rho \in C_0^\infty(R^n)$, $\rho \ge 0$, $\int \rho = 1$, and for any g we define

$$g * \rho_\varepsilon(x) = \int \rho_\varepsilon(x-y)g(y)dy, \quad \varepsilon > 0,$$

where $\rho_\varepsilon(y) = \rho(y/\varepsilon)/\varepsilon^n$. For any $r > 0$, $x_0 \in R^n$, let $B_r(x_0) = \{x \in R : |x - x_0| < r\}$. We will also assume $1 < p < m$, and let $u^{(m)}$ be the solution of (0.1) for the rest of the paper.

1.

In this section we will recall and extend some results of [BC], [BBC], [KPV] and [S2]. We will also prove some technical lemmas that will be used in the proof of the main theorem (Theorem 2.5) in section 2. We first recall a result of [KPV].

Theorem 1.1 (cf. Lemma 2.4 of [KPV]). *If $u_1^{(m)}, u_2^{(m)} \in L^\infty([0,1); L^1(R^n)) \cap L^\infty(R^n \times (0,1)) \cap C(R^n \times (0,1))$ are solutions of (0.1) in $R^n \times (0,1)$ with initial values $f_1, f_2 \in L^1(R^n) \cap L^\infty(R^n)$ respectively, $f_1, f_2 \geq 0$, then*

$$\text{(i)} \qquad \int_{R^n} (u_1^{(m)} - u_2^{(m)})_+(x,t)dx \leq \int_{R^n} (f_1 - f_2)_+(x)dx,$$

$$\text{(ii)} \qquad \int_{R^n} |u_1^{(m)} - u_2^{(m)}|(x,t)dx \leq \int_{R^n} |f_1 - f_2|(x)dx$$

for all $0 < t < 1$. Hence $u_1^{(m)} \leq u_2^{(m)}$ if $f_1 \leq f_2$. In particular the solution of (0.1) in $R^n \times (0,1)$ with initial value in $L^1(R^n) \cap L^\infty(R^n)$ is unique in the class $L^\infty([0,1); L^1(R^n)) \cap L^\infty(R^n \times (0,1)) \cap C(R^n \times (0,1))$.

Theorem 1.2. *(0.1) has a unique solution*

$$u^{(m)} \in L^\infty([0,1); L^1(R^n)) \cap L^\infty(R^n \times (0,1)) \cap C(R^n \times (0,1))$$

with

$$\text{(i)} \qquad \int u^{(m)}(x,t)dx \leq \int f dx \quad \forall 0 < t < 1,$$

$$\text{(ii)} \qquad \|u^{(m)}\|_{L^\infty(R^n \times (0,1))} \leq \|f\|_{L^\infty(R^n)}.$$

Proof. The proof is similar to the proof of Theorem 1.3 of [H3]. We refer the reader to [H3] for the details. □

Lemma 1.3. *Let $0 \leq f \leq M$ with $\operatorname{supp} f \subset B_R(0)$. Then the following hold.*

(i) For any $\delta > 0$ and $0 < T_1 < 1$ there exists $m' > p$ depending only on M, R and T_1 such that

$$u^{(m)}(x,t) \leq 1 + \delta$$

and

$$(u^{(m)})^m(x,t) \leq \delta$$

for all $x \in R^n$, $T_1 \leq t < 1$, $m \geq m'$.

(ii) There exist $R_1 > R$ and $m' > p$ depending only on M, R such that

$$\text{(1.1)} \qquad u^{(m)}(x,t) \equiv 0 \quad \forall |x| \geq R_1, 0 \leq t < 1, m \geq m'.$$

Proof. The proof is similar to the proof of Lemma 2.1 of [S2]. We let

$$w(x,t) = \frac{1}{(t+t_0)^{nk}} \left\{ a^2 - \frac{C_1|x|^2}{(t+t_0)^{2k}} \right\}_+^{1/m-1}$$

be the Barenblatt solution of $u_t = \Delta u^m$ [HP], where

$$a^2 = C_1 t_0^{-2k} R^2 + M^{m-1} t_0^{n(m-1)k},$$

$k=1/(n(m-1)+2)$, $C_1=(m-1)/(2m(n(m-1)+2))$, $t_0=C_1R^2/M^{m-1}$. Since $u^{(m)}$ is a subsolution of $u_t=\Delta u^m$ and $u^{(m)}(x,0)\le M\le w(x,0)$, by the maximum principle,

$$u^{(m)}(x,t)\le w(x,t)\quad \forall x\in R^n, 0\le t<1.$$

Hence

$$\begin{aligned}
u^{(m)}(x,t) &\le \frac{a^{2/m-1}}{(T_1+t_0)^{n/(n(m-1)+2)}}\\
&\le \frac{1}{T_1^{n/(n(m-1)+2)}}\cdot\left(\frac{2C_1R^2}{t_0^{2/(n(m-1)+2)}}\right)^{1/m-1}\\
&\le \frac{2^{1/m-1}(C_1R^2)^{n/(n(m-1)+2)}M^{2/(n(m-1)+2)}}{T_1^{n/(n(m-1)+2)}} \qquad (1.2)\\
&\to 1 \quad \text{as } m\to\infty \qquad \forall x\in R^n, T_1\le t<1.
\end{aligned}$$

By (1.2),

$$\begin{aligned}
u^{(m)m}(x,t) &\le \frac{2^{m/m-1}(C_1R^2)^{nm/(n(m-1)+2)}M^{2m/(n(m-1)+2)}}{T_1^{nm/(n(m-1)+2)}}\\
&\to 0 \quad \text{as } m\to\infty \qquad \forall x\in R^n, T_1\le t<1.
\end{aligned}$$

Hence (i) follows. We next observe that for each $0\le t<1$,

$$\operatorname{supp} u^{(m)}(\cdot,t)\subset \operatorname{supp} w(\cdot,t)\subset B_{R_t}(0)$$

where

$$\begin{aligned}
R_t &= \frac{a(t+t_0)^{1/(n(m-1)+2)}}{C_1^{1/2}}\le \frac{2a}{C_1^{1/2}}\\
&\le \frac{4R}{t_0^{1/(n(m-1)+2)}}\\
&\le \frac{4R}{(C_1R^2)^{1/(n(m-1)+2)}}\cdot M^{(m-1)/(n(m-1)+2)}\\
&\le 4RM^{1/n} \quad \text{as } m\to\infty.
\end{aligned}$$

Hence (ii) follows. □

Corollary 1.4. *Suppose f is as in Lemma 1.3. Let $\Omega\subset R^n$ be a bounded open set with $\partial\Omega\in C^2$, and let $\eta\in C^\infty(R^n\times(0,1))$. Then*

$$\int_{\tau_1}^{\tau_2}\int_\Omega u^{(m)m}\cdot\eta\,dxdt\to 0,\quad \int_{\tau_1}^{\tau_2}\int_{\partial\Omega} u^{(m)m}\frac{\partial\eta}{\partial n}d\sigma dt\to 0 \text{ as } m\to\infty$$

for any $0<\tau_1\le\tau_2<1$.

Proof. The corollary follows immediately from Lemma 1.3. □

Lemma 1.5. *Let $f\in C_0(R^n)$ and $h^{(m)}(x,t)=\int_0^t u^{(m)m}(x,\tau)d\tau$. Then $\{h^{(m)}\}_{m>p}$ is uniformly bounded on $R^n\times[0,1)$. For any sequence $\{h^{(m_i)}\}_{i=1}^\infty$, $m_i\to\infty$ as $i\to\infty$, of $\{h^{(m)}\}_{m>p}$, there exist a subsequence $\{h^{(m_i')}\}_{i=1}^\infty$ of $\{h^{(m_i)}\}_{i=1}^\infty$, a sequence of functions $\{h_k\}_{k=1}^\infty\subset L^\infty(R^n)$, $\widetilde{g}\in L^\infty(R^n)$, h_k, $\widetilde{g}\ge 0$,* $\operatorname{supp} h_k\subset B_{R_1}(0)$ *where*

R_1 is as in (ii) of Lemma 1.3, and a sequence $1 > \varepsilon_1 > \varepsilon_2 > \cdots > 0$, $\varepsilon_k \to 0$ as $k \to \infty$, such that

$$\begin{cases} h^{(m_i')}(\cdot, \varepsilon_k) \to h_k(\cdot) & \text{weakly in } (L^\infty(K))^* \text{ as } i \to \infty, \quad \forall k = 1, 2, \ldots, \\ h_k(\cdot) \to \widetilde{g}(\cdot) & \text{weakly in } (L^\infty(K))^* \text{ as } k \to \infty \end{cases} \tag{1.3}$$

for any compact subset $K \subset R^n$. Moreover, $\widetilde{g}$ satisfies

$$\begin{cases} \widetilde{g}(x) \le Nf(x) \le C(\|f\|_{L^1} + \|f\|_{L^\infty}) & \forall x \in R^n, \text{ for } n \ge 3, \\ |\nabla \widetilde{g}(x)| \le C(1 + \|f\|_{L^\infty}^{p-1})(\|f\|_{L^1} + \|f\|_{L^\infty}) & \forall x \in R^n, \text{ for } n = 1, 2 \end{cases} \tag{1.4}$$

and (0.5) for any measurable set $E \subset R^n$ where $C > 0$ is a constant independent of E and $m > p$, $N(x,y) = c_n|x-y|^{2-n}$, $c_n = 1/(n-2)|B_1(0)|$ for $n \ge 3$, and $Nf(x) = \int N(x,y)f(y)dy$.

Proof. We first observe that by Lemma 1.3 there exist $R_1 \ge 1$ and $m' > p$ such that (1.1) holds. For $n \in \mathcal{Z}^+$ and any $R > 0$, $\eta \in C(R^n)$, let G_R be the Green function for the ball $B_R(0)$ ($\Delta G_R = -\delta_0$),

$$G_R \eta(x) = \int_{B_R(0)} G_R(x,y)\eta(y)dy$$

and let

$$q_R(x,t) = \int_0^t u^{(m)m}(x,\tau)d\tau + G_R u^{(m)}(x,t) - G_R f(x) + \int_0^t G_R u^{(m)p}(x,\tau)d\tau.$$

Since $u^{(m)}$ is a solution of (0.1),

$$\int_{R^n} q(x,t)\Delta\eta(x)dx = 0 \quad \forall \eta \in C_0^\infty(B_R(0)), 0 \le t < 1$$

Hence for each $0 \le t < 1$ $q_R(\cdot,t)$ is harmonic in $B_R(0)$ and satisfies $q_R(x,t) \equiv 0$ for all $|x| = R$, $R \ge R_1$, by (1.1). By the maximum principle,

$$q_R(x,t) \equiv 0 \quad \forall |x| \le R, R \ge R_1, 0 \le t < 1$$

Thus

$$h^{(m)}(x,t) = \int_0^t u^{(m)m}(x,\tau)d\tau = -G_R u^{(m)}(x,t) + G_R f(x) - \int_0^t G_R u^{(m)p}(x,\tau)d\tau \tag{1.5}$$

for all $|x| \le R$, $R \ge R_1$, $0 \le t < 1$, $m > m'$. Since $G_R \ge 0$ and $\int G_R(x,y)dy = (R^2 - |x|^2)/2n$, we have by (1.5),

$$\begin{aligned} 0 \le h^{(m)}(x,t) \le & G_{R_1} f(x) \\ \le & \|f\|_{L^\infty} \cdot \int G_{R_1}(x,y)dy \\ \le & \left(\frac{R_1^2 - |x|^2}{2n}\right) \cdot \|f\|_{L^\infty} \\ \le & \frac{R_1^2}{2n}\|f\|_{L^\infty} \quad \forall |x| \le R_1, 0 \le t < 1, m > m'. \end{aligned}$$

Since $h^{(m)}(x,t)\equiv 0$ for all $|x|\ge R_1$, $0\le t\le 1$, $m>p$, and

$$h^{(m)}(x,t)\le(\|f\|_{L^\infty}+1)^{m'}\quad\forall x\in R^n,0\le t<1,m'\ge m>p.$$

$\{h^{(m)}\}_{m>p}$ is uniformly bounded on $R^n\times[0,1)$ for all $n\in\mathcal{Z}^+$. So any sequence $\{h^{(m_i)}\}_{i=1}^\infty$ of $\{h^{(q)}\}_{m>p}$ has a subsequence $\{h^{(m_{1,i})}\}_{i=1}^\infty$ such that $\{h^{(m_{1,i})}(\cdot,1/2)\}_{i=1}^\infty$ converges weakly in $(L^\infty(K))^*$ for any compact subset $K\subset R^n$. Similarly $\{h^{(m_{1,i})}\}_{i=1}^\infty$ has $\{h^{(m_{2,i})}\}_{i=1}^\infty$ such that $\{h^{(m_{2,i})}(\cdot,1/3)\}_{i=1}^\infty$ converges weakly in $(L^\infty(K))^*$ for any compact subset $K\subset R^n$. Repeating the argument $\{h^{(m_{k,i})}\}_{i=1}^\infty$ will have a subsequence $\{h^{(m_{k+1,i})}\}_{i=1}^\infty$ such that $\{h^{(m_{k+1,i})}(\cdot,1/(k+2))\}_{i=1}^\infty$ converges weakly in $(L^\infty(K))^*$ for any compact subset $K\subset R^n$ for all $k=0,1,\dots$, where $m_{0,i}=m_i$ for all $i\in\mathcal{Z}^+$.

The first part of the lemma then follows by a diagonalization argument as in the proof of Lemma 1.8 of [H3], and (1.3) holds. Since (1.3) holds, we may assume without loss of generality that for a.e. $x\in R^n$,

$$\begin{cases} h^{(m_i')}(x,\varepsilon_k)\to h_k(x) & \text{as } i\to\infty\quad\forall k=1,2,\dots,\\ h_k(x)\to\widetilde{g}(x) & \text{as } k\to\infty.\end{cases}$$

To prove (1.4) we observe that for $n\ge 3$, by letting $R\to\infty$ in (1.5), we have

$$\begin{aligned} h^{(m)}(x,t)&=-Nu^{(m)}(x,t)+Nf(x)-\int_0^t Nu^{(m)p}(x,\tau)d\tau\\ &\le Nf(x)\le C(\|f\|_{L^\infty}+\|f\|_{L^1})\end{aligned}$$

for all $x\in R^n$, $0\le t<1$, $m>m'$, where $C>0$ is a constant independent of f and m. The first inequality of (1.4) then follows by putting $m=m_i'$, $t=\varepsilon_k$ and letting $i\to\infty$ and $k\to\infty$. For the second inequality of (1.4), observe that for $n=2$ by (1.5) we have

$$\begin{aligned} \nabla h^{(m)}(x,t)&=-\nabla G_Ru^{(m)}(x,t)+\nabla G_Rf(x)-\int_0^t\nabla G_Ru^{(m)p}(x,\tau)d\tau\\ &\to-\widetilde{N}u^{(m)}(x,t)+\widetilde{N}f(x)-\int_0^t\widetilde{N}u^{(m)p}(x,\tau)d\tau\quad\text{as } R\to\infty\\ \Rightarrow\quad|\nabla h^{(m)}(x,t)|&\le\frac{(1+\|f\|_{L^\infty}^{p-1})}{\pi}\cdot\left(\int_{R^2}\frac{1}{|x-y|}u^{(m)}(y,t)dy+\int_{R^2}\frac{1}{|x-y|}f(y)dy\right.\\ &\qquad\left.+\int_0^t\int_{R^2}\frac{1}{|x-y|}u^{(m)}(y,\tau)dyd\tau\right)\end{aligned}\tag{1.6}$$

for all $x\in R^2$, $0\le t<1$, $m>m'$ by Theorem 1.2, where

$$\widetilde{N}=(\widetilde{N}_1,\widetilde{N}_2)=\frac{-1}{2\pi}\cdot\frac{x-y}{|x-y|^2}$$

and

$$\widetilde{N}\eta=\frac{-1}{2\pi}\cdot\int_{R^2}\frac{x-y}{|x-y|^2}\eta(y)dy$$

for any function η of R^2. Since

$$\begin{aligned}\int_{R^2}\frac{1}{|x-y|}u^{(m)}(y,t)dy&=\left(\int_{|x-y|\le1}+\int_{|x-y|>1}\right)\frac{1}{|x-y|}u^{(m)}(y,t)dy\\ &\le C(\|f\|_{L^\infty}+\|f\|_{L^1})\quad\forall x\in R^2,0\le t<1,\end{aligned}$$

where $C>0$ is a constant independent of f and $m>p$, we have

$$|\nabla h^{(m)}(x,t)| \le C(1+\|f\|_{L^\infty}^{p-1})(\|f\|_{L^\infty}+\|f\|_{L^1}) \quad \forall x\in R^2, 0\le t<1, \tag{1.7}$$

where $C>0$ is a constant independent of f and $m>p$. Hence for $n=2$, $\{\nabla h^{(m)}(x,t)\}_{m>p}$ is uniformly bounded on $R^n\times[0,1)$. By a diagonalization argument there exist a subsequence $\{\varepsilon_{k_j}\}$ of $\{\varepsilon_k\}$ and a subsequence $\{m_i''\}$ of $\{m_i'\}$ such that

$$\nabla h^{(m_i'')}(x,\varepsilon_{k_j}) \to \nabla h_{k_j}(x) \quad \text{weakly in } L^\infty(K)^* \text{ and a.e. } x\in R^2 \quad \text{as } i\to\infty$$

for all $j=1,2,\dots$ and

$$\nabla h_{k_j}(x)\to\nabla\widetilde{g}(x) \quad \text{weakly in } L^\infty(K)^* \text{ and a.e. } x\in R^2 \quad \text{as } j\to\infty$$

for any compact subset $K\subset R^2$. Putting $m=m_i''$, $t=\varepsilon_{k_j}$ in (1.7) and letting $i\to\infty$, $j\to\infty$, we get the second inequality of (1.4) for $n=2$.

For $n=1$, since $u^{(m)}$ satisfies (0.1),

$$\begin{aligned}&\int_R h^{(m)}(x,t)\eta''(x)dx-\int_R\left(\int_0^t u^{(m)p}(x,\tau)d\tau\right)\eta(x)dx\\ &=\int_R u^{(m)}(x,t)\eta(x)dx-\int_R f\eta dx \quad \forall\eta\in C_0^\infty(R)\\ &\Rightarrow \quad h^{(m)''}(\cdot,t)=h_1^{(m)}(\cdot,t)+u^{(m)}(\cdot,t)-f \quad \text{in } \mathcal{D}'(R) \quad \forall 0\le t<1\end{aligned} \tag{1.8}$$

where

$$h_1^{(m)}(x,t)=\int_0^t u^{(m)p}(x,\tau)d\tau.$$

Since $u^{(m)}\in C_0(R)$ by (1.1),

$$h_1^{(m)}(\cdot,t)+u^{(m)}(\cdot,t)-f\in C_0(R^n).$$

Thus $h^{(m)}(\cdot,t)\in C_0^2(R)$ for all $0\le t<1$, $m>m'$. For any $x_0\in R$, $x_0\ge -R_1$, let

$$\eta(x)=\zeta(-R_1-2-x)\zeta(\frac{x-x_0}{\varepsilon})$$

where $\zeta\in C^\infty(R)$ satisfies $0\le\zeta\le1$, $\zeta(s)=1$ if $s\le-1$, $\zeta=0$ if $s\ge0$, $\zeta'\le0$, $\zeta'(-1)=\zeta'(0)=0$. Then $0\le\eta\le1$, $\eta(x)\equiv1$ for $-R_1-1<x<x_0-\varepsilon$, and $\eta(x)\equiv0$ for $x>x_0$ or $x<-R_1-2$. Putting η into (1.8) and integrating by parts,

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{x_0-\varepsilon}^{x_0} h^{(m)'}(x,t)\zeta'((x-x_0)/\varepsilon)dx+\int_R\left(\int_0^t u^{(m)p}(x,\tau)d\tau\right)\eta(x)dx\\ &=-\int_R u^{(m)}(x,t)\eta(x)dx+\int_R f\eta dx.\end{aligned} \tag{1.9}$$

Since $h^{(m)}(\cdot,t)\in C_0^2(R)$,

$$\begin{aligned}&\left|\frac{1}{\varepsilon}\int_{x_0-\varepsilon}^{x_0} h^{(m)'}(x,t)\zeta'((x-x_0/\varepsilon))dx+h^{(m)'}(x_0,t)\right|\\ &\le\frac{1}{\varepsilon}\int_{x_0-\varepsilon}^{x_0}|h^{(m)'}(x,t)-h^{(m)'}(x_0,t)||\zeta'((x-x_0)/\varepsilon)|dx\\ &\le\sup_{x_0-\varepsilon\le x\le x_0}|h^{(m)'}(x,t)-h^{(m)'}(x_0,t)|\to0 \quad \text{as } \varepsilon\to0.\end{aligned}$$

Letting $\varepsilon \to 0$ in (1.9), we get

$$h^{(m)\prime}(x_0,t) = \int_{-\infty}^{x_0} [h_1^{(m)}(x,t) + u^{(m)}(x,t) - f(x)]dx \qquad \forall x_0 \in R, 0 \le t < 1, m > m'$$

$$\Rightarrow \quad |h^{(m)\prime}(x,t)| \le C' = (2 + \|f\|_{L^\infty}^{p-1})\|f\|_{L^1} < \infty \quad \forall x \in R, 0 \le t < 1, m > m'. \tag{1.10}$$

Hence $\{h^{(m)\prime}(x,t)\}_{m>p}$ is uniformly bounded on $R \times [0,1)$. The second inequality of (1.4) for $n = 1$ then follows by the same argument as in the case $n = 2$. We next observe that since

$$\int_E \frac{1}{|x-y|}dy = \left(\int_{E\cap B_{R_0}(0)} + \int_{E\cap B_{R_0}(0)^c}\right)\frac{1}{|x-y|}dy \le \int_{B_{R_0}(0)} \frac{1}{|x-y|}dy = 2|B_1(0)|^{1/2}|E|^{1/2}$$

for any measurable set $E \subset R^2$ where $R_0 > 0$ is such that $|B_{R_0}(0)| = |E|$, we have

$$\int_E \left(\int_{R^2} \frac{1}{|x-y|}u^{(m)}(y,t)dy\right)dx = \int_{R^2}\left(\int_E \frac{1}{|x-y|}dx\right)u^{(m)}(y,t)dy \le 2|B_1(0)|^{1/2}\|f\|_{L^1}|E|^{1/2}.$$

Hence by (1.6),

$$\int_E |\nabla h^{(m)}(x,t)|dx \le \frac{6(1+\|f\|_{L^\infty}^{p-1})}{\pi}|B_1(0)|^{1/2}\|f\|_{L^1}|E|^{1/2}$$

for any measurable set $E \subset R^2$. If we put $m = m_i''$, $t = \varepsilon_{k_j}$ in the above inequality and let $i \to \infty$, $j \to \infty$, the second inequality of (0.5) then follows by Fatou's Lemma. Similarly the first inequality of (0.5) follows by integrating the first inequality of (1.4). The lemma is proved. □

We next recall a result of [BBC] and a result of [BC].

Theorem 1.6 (cf. [BBC])**.** *For any $0 \le f \in L^1(R^n)$, there exist at most one function $g \in L^1$, $0 \le g \le 1$, $\|g\|_{L^1} \le \|f\|_{L^1}$, and at most one function $\widetilde{g} \in L^1_{loc}(R^n)$, $\widetilde{g} \ge 0$, satisfying (0.3),(0.4),(0.5).*

Theorem 1.7. *There exists a constant $C > 0$ depending only on $\|f\|_{L^1}$, $\|f\|_{L^\infty}$, p, and independent of $m > p$, such that*

$$\int_{R^n} |u^{(m)}(x,t+h) - u^{(m)}(x,t)|dx \le \frac{Ch}{t}$$

for all $0 < t \le t+h < 1$, $0 \le h \le t/2$. Hence $u^{(m)} \in C((0,1);L^1(R^n))$.

Proof. The theorem follows directly from the proof of Theorem 4 and Theorem 7 of [BC]. □

2.

In this section we will use an adaptation of the argument in [H3] to prove the main theorem. We will first state a preliminary version of the main theorem.

Theorem 2.1. *Let $f \in C_0(R^n)$, $f \ge 0$. For any subsequence $\{u^{(m_i)}\}_{i=1}^\infty$, $m_i \to \infty$ as $i \to \infty$, of $\{u^{(m)}\}_{m>p}$, there exist a subsequence $\{u^{(m_i')}\}_{i=1}^\infty$ of $\{u^{(m_i)}\}_{i=1}^\infty$ and a $u^{(\infty)} \in C((0,1); L^1(R^n))$, $0 \le u^{(\infty)} \le 1$, such that*

$$u^{(m_i')} \to u^{(\infty)} \text{ in } L^1_{loc}(R^n \times (0,1)) \quad \text{as } i \to \infty.$$

Moreover $u^{(\infty)}$ satisfies $v_t = -v^p$ in $\mathcal{D}'(R^n \times (0,1))$ with initial value $g \in L^1(R^n)$, $0 \le g \le 1$, $\|g\|_{L^1} = \|f\|_{L^1}$, satisfying $g - \Delta\widetilde{g} = f$ for some function $\widetilde{g} \in L^\infty(R^n)$, $\widetilde{g} \ge 0$, satisfying (0.5) and (1.4).

Proof. The proof is similar to the proof of Theorem 2.1 of [H3] and Theorem 1 of [S2]. Since by Theorem 1.1,

$$\int_{R^n} |u^{(m)}(x+x_0,t) - u^{(m)}(x,t)|dx \le \int_{R^n} |f(x+x_0) - f(x)|dx \quad \forall x_0 \in R^n, 0 \le t < 1$$

by Lemma 1.3(ii) and Theorem 1.7 $\{u^{(m)}\}_{m>p}$ is precompact in $L^1_{loc}(R^n \times (0,1))$. Hence there exist a function $u^{(\infty)} \in L^1(R^n \times (0,1))$ and a subsequence $\{u^{(m_i')}\}_{i=1}^\infty$ of $\{u^{(m_i)}\}_{i=1}^\infty$ such that $\{u^{(m_i')}\}_{i=1}^\infty$ converges to $u^{(\infty)}$ in $L^1_{loc}(R^n \times (0,1))$ as $i \to \infty$. Without loss of generality we may assume that $u^{(m_i)} \to u^{(\infty)}$ in $L^1_{loc}(R^n \times (0,1))$ and for a.e. $(x,t) \in R^n \times (0,1)$ as $i \to \infty$. Then $0 \le u^{(\infty)} \le 1$ by Lemma 1.3.

We claim that $u^{(\infty)} \in C((0,1); R^n)$. To prove the claim we observe first that by Theorem 1.7, $\forall 0 < \tau_1 < \tau_2 < 1$, there exists $C > 0$ such that

$$\int_{R^n} |u^{(m_i)}(x,s') - u^{(m_i)}(x,s)|dx \le C(s'-s) \quad \forall \tau_1 < s < s' < \tau_2, 0 < s'-s < s/2, i \in \mathcal{Z}^+. \tag{2.1}$$

For any t, t' satisfying $\tau_1 < t < t' < \tau_2$, $0 < t'-t < t/4$, we let

$$h_0 = \min\left(t - \tau_1, \tau_2 - t', \frac{t'-t}{2}, \frac{3t-2t'}{5}\right).$$

Note that $3t - 2t' > 3t - 2(5t/4) = t/2 > 0$. Then for any $0 < h_1$, $h_2 < h_0$, $s \in (t-h_1, t+h_1)$, $s' \in (t'-h_2, t'+h_2)$, we have $\tau_1 < s < s' < \tau_2$,

$$s' - s > t' - h_2 - (t+h_1) \ge t' - t - 2h_0 \ge 0$$

and

$$s' - s < t' + h_2 - (t - h_1) \le t' - t + 2h_0 \le t' - t + 2\left(\frac{3t-2t'}{5}\right) = \frac{t+t'}{5}$$

$$\Rightarrow \quad \frac{s}{2} > \frac{t-h_1}{2} \ge \frac{t - \frac{3t-2t'}{5}}{2} = \frac{t+t'}{5} \ge s' - s.$$

Hence (2.1) holds for all $|s-t|<h_1$, $|s'-t'|<h_2$. By integrating (2.1) over $|s-t|<h_1$, $|s'-t'|<h_2$, we have

$$\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n}|u^{(m_i)}(x,s')-u^{(m_i)}(x,s)|dxdsds' \leq C\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}(s'-s)dsds'.$$

Hence for all $\tau_1<t<t'<\tau_2$, $0<t'-t<t/4$, $0<h_1$, $h_2<h_0$,

$$\begin{aligned}
&\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n}|u^{(\infty)}(x,s')-u^{(\infty)}(x,s)|dxdsds'\\
\leq&\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n}|u^{(\infty)}(x,s')-u^{(m_i)}(x,s')|dxdsds'\\
&+\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n}|u^{(m_i)}(x,s')-u^{(m_i)}(x,s)|dxdsds'\\
&+\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n}|u^{(m_i)}(x,s)-u^{(\infty)}(x,s)|dxdsds'\\
\leq&2h_1\int_{t'-h_2}^{t'+h_2}\int_{R^n}|u^{(\infty)}(x,s')-u^{(m_i)}(x,s')|dxds'\\
&+C\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}(s'-s)dsds'\\
&+2h_2\int_{t-h_1}^{t+h_1}\int_{R^n}|u^{(m_i)}(x,s)-u^{(\infty)}(x,s)|dxds.
\end{aligned}$$

Since by Lemma 1.3, there exists $R_1>0$ and $m'>p$ such that (1.1) holds, it follows that $u^{(m_i)}\to u^{(\infty)}$ in $L^1(R^n\times(\tau_1,\tau_2))$ as $i\to\infty$. Hence by letting $i\to\infty$, we get

$$\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n}|u^{(\infty)}(x,s')-u^{(\infty)}(x,s)|dxdsds'\leq C\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}(s'-s)dsds'$$

for all $\tau_1<t<t'<\tau_2$, $0<t'-t<t/4$, $0<h_1$, $h_2<h_0$. Dividing both side by $4h_1h_2$ and letting first $h_1\to 0$ and then $h_2\to 0$, we get, by the Lebesgue differentiation theorem [St],

$$\int_{R^n}|u^{(\infty)}(x,t')-u^{(\infty)}(x,t)|dx\leq C(t'-t) \tag{2.2}$$

for a.e. t', t, $\tau_1<t<t'<\tau_2$, $0<t'-t<t/4$. By redefining $u^{(\infty)}$ on a set of measure zero if necessary, we can assume without loss of generality that (2.1) holds for all t', t satisfying $\tau_1<t<t'<\tau_2$, $0<t'-t<t/4$. Hence $u^{(\infty)}\in C((0,1);L^1(R^n))$. Putting $h(u)=-u^p$, $u=u^{(m_i)}$ in (0.7) and letting $i\to\infty$, we find that $u^{(\infty)}$ satisfies

$$\int_0^1\int_\Omega(v\eta_t-v^p\eta)dxd\tau=0 \tag{2.3}$$

for any bounded open set $\Omega\subset R^n$ with $\partial\Omega\in C^2$, $\eta\in C^\infty(\Omega\times[0,1))$, $\eta\equiv 0$ on $\partial\Omega\times[0,1)$ and $\eta(\cdot,t)\equiv 0$ for t close to 0 and 1. Since $u^{(\infty)}\in C((0,1);L^1(R^n))$, by putting

$$\eta(x,t)=\psi(x,t)\zeta(\frac{t-\tau_2}{\varepsilon})\zeta(\frac{\tau_1-t}{\varepsilon})$$

into (2.3), where ζ is as in the proof of Lemma 1.5, $0 < \tau_1 < \tau_2 < 1$, $\psi \in C^\infty(\Omega \times [0,1))$, $\psi \equiv 0$ on $\partial\Omega \times [0,1)$, and letting $\varepsilon \to 0$, by the same argument as the proof of Theorem 3.2 of [DKe] we get that $u^{(\infty)}$ satisfies

$$\int_\Omega v\psi dx\Big|_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2}\int_\Omega (v\psi_t - v^p\psi)dxd\tau \tag{2.4}$$

for any bounded open set $\Omega \subset R^n$ with $\partial\Omega \in C^2$, $\psi \in C^\infty(\Omega \times [\tau_1,\tau_2])$, $\psi \equiv 0$ on $\partial\Omega \times [\tau_1,\tau_2]$.

To show that g satisfies (0.3) we let $h^{(m)}$ be as in Lemma 1.5 and Ω be a bounded open subset of R^n with $\partial\Omega \in C^2$. Then by Lemma 1.5 there exists a constant $C_1 > 0$ such that $\|h^{(m)}\|_{L^\infty(R^n\times[0,1))} \le C_1$ for all $m > p$, and there exist a subsequence $\{h^{(m_i')}\}_{i=1}^\infty$ of $\{h^{(m_i)}\}_{i=1}^\infty$, a sequence of functions $\{h_k\}_{k=1}^\infty \subset L^\infty(R^n)$, $\widetilde{g} \in L^\infty(R^n)$, h_k, $\widetilde{g} \ge 0$, and a sequence $1 > \varepsilon_1 > \varepsilon_2 > \cdots > 0$, $\varepsilon_k \to 0$ as $k \to \infty$, such that (1.3),(1.4), and (0.5) hold.

Since by Theorem 1.2,

$$\int_{R^n} u^{(m_i)}(x,\varepsilon_k)dx \le \int_{R^n} f(x)dx, \quad \forall i,k = 1,2,\ldots,$$

letting $i \to \infty$, we get by Fatou's Lemma,

$$\int_{R^n} u^{(\infty)}(x,\varepsilon_k)dx \le \int_{R^n} f(x)dx, \quad \forall k = 1,2,\ldots.$$

Hence there exists a subsequence $\{u^{(\infty)}(x,\varepsilon_k')\}_{k=1}^\infty$ of $\{u^{(\infty)}(x,\varepsilon_k)\}_{k=1}^\infty$ such that $u^{(\infty)}(x,\varepsilon_k') \to d\mu$ weakly in $L^1(R^n)$ for some measure $d\mu \in L^1(R^n)$. Since $0 \le u^{(\infty)} \le 1$, we have $d\mu = gdx$ for some function $g \in L^1(R^n)$ with $0 \le g \le 1$, $\|g\|_{L^1} \le \|f\|_{L^1}$. We may assume without loss of generality that $u^{(\infty)}(x,\varepsilon_k) \to g$ weakly in $L^1(R^n)$ as $k \to \infty$. Putting $\tau_1 = \varepsilon_k$ in (2.4) and letting first $k \to \infty$ and then $\tau_2 \to 0$, we get

$$\lim_{t\to 0}\int u^{(\infty)}\eta dx = \int g\eta dx \quad \forall \eta \in C_0^\infty(R^n).$$

Putting $h(u) = -u^p$, $u = u^{(m)}$, $\tau_2 = \varepsilon_k$, $m = m_i'$ in (0.7) and letting first $\tau_1 \to 0$ and then $i \to \infty$, we get by Corollary 1.4 and Lemma 1.5,

$$\begin{aligned}\int_{R^n} h_k(x)\Delta\eta(x)dx - \int_0^{\varepsilon_k}\int_{R^n} u^{(\infty)p}(x,\tau)\eta(x)dxd\tau \\ = \int_{R^n} u^{(\infty)}(x,\varepsilon_k)\eta(x)dx - \int_{R^n} f\eta dx\end{aligned} \tag{2.5}$$

for all $\eta \in C_0^\infty(R^n)$, $k = 1,2,\ldots$. Letting $k \to \infty$, we get by Lemma 1.5,

$$g - \Delta\widetilde{g} = f \text{ in } \mathcal{D}'(R^n)$$

for some function $\widetilde{g} \in L^\infty(R^n)$, $\widetilde{g} \ge 0$, satisfying (1.4). Let $R_1 > 0$ be as in (ii) of Lemma 1.3 and let $\psi \in C_0^\infty(R^n)$, $0 \le \psi \le 1$, be such that $\psi(x) = 1$ for $|x| \le R_1 + 1$, $\psi(x) = 0$ for $|x| \ge R_1 + 2$. Since $\operatorname{supp} h_k \subset B_{R_1}(0)$ by Lemma 1.5 and

$$\operatorname{supp} u^{(m)} \subset B_{R_1}(0) \quad \Rightarrow \quad \operatorname{supp} u^{(\infty)} \subset \overline{B_{R_1}(0)}$$

putting $\eta = \psi$ in (2.5) we have

$$\int_0^{\varepsilon_k}\int_{R^n} u^{(\infty)p}(x,\tau)dxd\tau = \int_{R^n} u^{(\infty)}(x,\varepsilon_k)dx - \int_{R^n} fdx$$
$$\Rightarrow \quad \int_{R^n} gdx = \int_{R^n} fdx \quad \text{as } k\to\infty.$$

This completes the proof of Theorem 2.1. □

Lemma 2.2. *If $u^{(\infty)}$ and g is as in Theorem 2.1, then for a.e. $x \in R^n$,*

$$u^{(\infty)}(x,t) - g(x) = -\int_0^t u^{(\infty)p}(x,s)ds \quad \forall 0<t<1 \tag{2.6}$$

and

$$\lim_{t\to 0} u^{(\infty)}(x,t) = g(x). \tag{2.7}$$

Proof. Putting $\psi(x,t) = \rho_\varepsilon(y-x)$ in (2.4) and letting first $\tau_1 \to 0$ and then $\varepsilon \to 0$, we get (2.6). Since $0 \le u^{(\infty)} \le 1$, (2.7) is a direct consequence of (2.6). □

Lemma 2.3. *Let f, g, $\widetilde{g}$ be as in Theorem 2.1. Then (0.4) holds.*

Proof. Let $u^{(m_i')}$, $u^{(\infty)}$ be as in Theorem 2.1, and let $v^{(m)}$ be the solution of

$$\begin{cases} v_t = \Delta v^m, v\ge 0, & \text{in } R^n\times(0,1),\\ v(x,0) = f(x) & \forall x\in R^n.\end{cases}$$

By the result of [BBH], there exists $g_1 \in L^1(R^n)$, $0 \le g_1 \le 1$, $\widetilde{g}_1 \ge 0$, satisfying

$$\begin{cases} g_1 - \Delta\widetilde{g}_1 = f & \text{in } \mathcal{D}'(R^n),\\ \widetilde{g}_1(x) = 0 & \text{whenever } g_1(x)<1 \text{ a.e. } x\in R^n,\\ \int_{R^n} g_1 = \int_{R^n} f \end{cases}$$

and $v^{(m)}(x,t) \to g_1(x)$ in $L^1(R^n)$ uniformly in t on compact subsets of $(0,1)$ as $m\to\infty$. Since $u^{(m)}$ satisfies $u_t \le \Delta u^m$, by the maximum principle,

$$u^{(m)}(x,t) \le v^{(m)}(x,t) \quad \forall (x,t)\in R^n\times(0,1)$$
$$\Rightarrow \quad u^{(\infty)}(x,t) \le g_1(x) \quad \text{a.e. } x\in R^n, 0<t<1 \text{ as } m=m_i'\to\infty$$
$$\Rightarrow \quad g(x) \le g_1(x) \quad \text{a.e. } x\in R^n \text{ as } t\to 0.$$

Since $\int_{R^n} gdx = \int_{R^n} fdx = \int_{R^n} g_1dx$, $g(x) = g_1(x)$ a.e. $x\in R^n$. Thus $\widetilde{g} = \widetilde{g}_1$ and the lemma follows. □

As a consequence of Theorem 1.6, Theorem 2.1, Lemmas 2.2, 2.3, and the uniqueness of a solution of the ODE (2.6),(2.7), we have

Theorem 2.4. *Suppose $f\in C_0(R^n)$, $f\ge 0$. Then there exists a unique function $u^{(\infty)} \in C((0,1);R^n)$, $0\le u^{(\infty)}\le 1$, such that $u^{(m)} \to u^{(\infty)}$ in $L^1_{loc}(R^n\times(0,1))$ as $m\to\infty$. Moreover $u^{(\infty)}$ satisfies (0.2) with initial value $g\in L^1(R^n)$, $0\le g\le 1$, $\|g\|_{L^1(R^n)} = \|f\|_{L^1(R^n)}$, satisfying $g - \Delta\widetilde{g} = f$ for some function $\widetilde{g}\in L^\infty(R^n)$, $\widetilde{g}\ge 0$, satisfying (0.4),(0.5), and (1.4).*

We are now ready to state and prove the main theorem.

Theorem 2.5. *For any $f \in L^1(R^n) \cap L^\infty(R^n)$, $f \geq 0$, there exists a unique function $u^{(\infty)} \in C((0,1);R^n)$, $0 \leq u^{(\infty)} \leq 1$, such that $u^{(m)}$ converges to $u^{(\infty)}$ in $L^1_{loc}(R^n \times (0,1))$ as $m \to \infty$. Moreover $u^{(\infty)}$ satisfies (0.2) with initial value $g \in L^1(R^n)$, $0 \leq g \leq 1$, $\|g\|_{L^1(R^n)} \leq \|f\|_{L^1(R^n)}$, satisfying $g - \Delta\widetilde{g} = f$ for some function $\widetilde{g} \in L^\infty_{loc}(R^n)$, $\widetilde{g} \geq 0$, satisfying (0.4) and (0.5).*

Proof. The proof is a modification of the proof of Theorem 2.10 of [H3]. We first choose a sequence $\{f_j\}_{j=1}^\infty \subset C_0(R^n)$ such that $\|f_j\|_{L^\infty(R^n)} \leq \|f\|_{L^\infty(R^n)} + 1$, $\|f_j\|_{L^1(R^n)} \leq \|f\|_{L^1(R^n)} + 1$ for all $j = 1, 2, \dots$, and $\|f_j - f\|_{L^1(R^n)} \to 0$ as $j \to \infty$.

For all $j = 1, 2, \dots$, let $u_j^{(m)}$ be the solution of (0.1) in $R^n \times (0,1)$ with initial value $u_j^{(m)}(x,0) = f_j(x)$ and let $u^{(\infty)}$ be the unique function given by Theorem 2.4 such that $u_j^{(m)} \to u_j^{(\infty)}$ in $L^1_{loc}(R^n \times (0,1))$ as $m \to \infty$. Then $u_j^{(\infty)}$ satisfies (0.2) with initial value $g_j \in L^1(R^n)$, $0 \leq g_j \leq 1$, satisfying

(2.8)
$$\begin{cases} \displaystyle\int_{R^n} g_j \leq \int_{R^n} f_j \leq \int_{R^n} f\,dx + 1, \\ g_j - \Delta\widetilde{g}_j = f \quad \text{ in } \mathcal{D}'(R^n) \text{ for some } \widetilde{g}_j \in L^\infty(R^n), \widetilde{g}_j \geq 0, \text{ satisfying (1.4), (0.5)}, \\ \widetilde{g}_j(x) = 0 \quad \text{ whenever } g_j(x) < 1 \text{ a.e. } x \in R^n \end{cases}$$

for all $j = 1, 2, \dots$. As in the proof of Theorem 2.1, by Theorems 1.1 and 1.7 $\{u^m\}_{m>p}$ is precompact in $L^1_{loc}(R^n \times (0,1))$. Thus any subsequence $\{u^{(m_i)}\}_{i=1}^\infty$, $m_i \to \infty$ as $i \to \infty$, of $\{u^{(m)}\}_{m>p}$ has a subsequence $\{u^{(m_i')}\}_{i=1}^\infty$ such that $u^{(m_i')} \to u^{(\infty)}$ in $L^1_{loc}(R^n \times (0,1))$ and $u^{(m_i')}(x,t) \to u^{(\infty)}(x,t)$ a.e. $(x,t) \in R^n \times (0,1)$ as $i \to \infty$ for some function $u^{(\infty)} \in L^1(R^n \times (0,1))$. By Theorem 1.1,

$$\begin{aligned} &\int_{R^n} |u_j^{(m_i')} - u^{(m_i')}|(x,t)dx \leq \int_{R^n} |f_j - f|(x)dx \qquad \forall i, j = 1, 2, \dots \\ (2.8) \qquad &\Rightarrow \quad \int_{\tau_1}^{\tau_2}\int_{R^n} |u_j^{(m_i')} - u^{(m_i')}|(x,t)dxdt \\ &\leq (\tau_2 - \tau_1)\int_{R^n} |f_j - f|(x)dx \quad \forall 0 < \tau_1 \leq \tau_2 < 1. \end{aligned}$$

Letting $i \to \infty$, we get by Fatou's Lemma,

(2.10)
$$\int_{\tau_1}^{\tau_2}\int_{R^n} |u_j^{(\infty)} - u^{(\infty)}|(x,t)dxdt \leq (\tau_2 - \tau_1)\int_{R^n} |f_j - f|(x)dx \to 0 \text{ as } j \to \infty$$

for all $0 < \tau_1 \leq \tau_2 < 1$.

Hence $u^{(\infty)}$ is the limit of the functions $\{u_j^{(\infty)}\}_{j=1}^\infty$ in $L^1_{loc}(R^n \times (0,1))$ as $j \to \infty$. Thus $u^{(\infty)}$ is unique and $0 \leq u^{(\infty)} \leq 1$, since $0 \leq u_j^{(\infty)} \leq 1$ for all $j = 1, 2, \dots$.

Putting $v = u_j^{(\infty)}$ in (2.3) and letting $j \to \infty$, we see that $u^{(\infty)}$ satisfies (2.3). We next claim that $u^{(\infty)} \in C((0,1);R^n)$. To prove the claim we first observe that by the proof of Theorem 2.1 there exists a constant $C > 0$ independent of $j = 1, 2, \dots$ such that (2.2) holds for $u_j^{(\infty)}$ for all $j = 1, 2, \dots$. For any t, t' satisfying $0 < \tau_1 < t < t' < \tau_2 < 1$, $0 < t' - t < t/8$, we let

$$h_0 = \min\left(t - \tau_1, \tau_2 - t', \frac{t' - t}{2}, \frac{5t - 4t'}{9}\right).$$

As in the proof of Lemma 1.5, for any $0 < h_1,\ h_2 < h_0,\ s \in (t-h_1, t+h_1)$, $s' \in (t'-h_2, t'+h_2)$, we have $\tau_1 < s < s' < \tau_2$, $0 < s'-s < s/4$. Hence for any t, t' satisfying $0 < \tau_1 < t < t' < \tau_2 < 1$, $0 < t'-t < t/8$, we have

$$
\begin{aligned}
&\int_{R^n} |u_j^{(\infty)}(x,s') - u_j^{(\infty)}(x,s)|dx \le C(s'-s)\\
&\qquad\qquad \forall |s-t| < h_1, |s'-t'| < h_2, j = 1,2,\dots\\
&\Rightarrow \int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n} |u_j^{(\infty)}(x,s') - u_j^{(\infty)}(x,s)|dxdsds'\\
&\le C\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}(s'-s)dsds' \quad \forall j = 1,2\dots\\
&\Rightarrow \int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n} |u^{(\infty)}(x,s') - u^{(\infty)}(x,s)|dxdsds'\\
&\le \int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n} |u^{(\infty)}(x,s') - u_j^{(\infty)}(x,s')|dxdsds'\\
&\quad + \int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n} |u_j^{(\infty)}(x,s') - u_j^{(\infty)}(x,s)|dxdsds'\\
&\quad + \int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}\int_{R^n} |u_j^{(\infty)}(x,s) - u^{(\infty)}(x,s)|dxdsds'\\
&\le C\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}(s'-s)dsds' + 8h_1h_2\|f_j - f\|_{L^1}\\
&\quad \to C\int_{t'-h_2}^{t'+h_2}\int_{t-h_1}^{t+h_1}(s'-s)dsds' \quad \text{ as } j\to\infty
\end{aligned}
$$

by (2.10) for some constant depending on $\|f\|_{L^1}$, $\|f\|_{L^\infty}$ and τ_1, τ_2. Dividing both sides by $4h_1h_2$ and letting first $h_1 \to 0$ and then $h_2 \to 0$, we get by Lebesgue's differentiation theorem that (2.2) holds for $u^{(\infty)}$, $\tau_1 < t < t' < \tau_2$, $0 < t'-t < t/8$. Hence $u^{(\infty)} \in C((0,1);L^1(R^n))$. By the same argument as the proof of Theorem 2.1, $u^{(\infty)}$ satisfies (2.4) and has an initial trace g, $0 \le g \le 1$; and $\|g\|_{L^1(R^n)} \le \|f\|_{L^1(R^n)}$ by Theorem 1.1.

We will next show that $g_j \to g$ in $L^1(R^n)$ as $j \to \infty$. Observe that, since both $u_j^{(\infty)}$ and $u^{(\infty)} \in C((0,1);L^1(R^n))$, dividing both side of (2.10) by $(\tau_2 - \tau_1)$ and letting $\tau_2 - \tau_1 \to 0$, we get

$$
(2.11)\quad \int_{R^n} |u_j^{(\infty)} - u^{(\infty)}|(x,t)dx \le \int_{R^n} |f_j - f|(x)dx \quad \forall 0 < t < 1, j = 1,2,\dots.
$$

Since for a.e. $x \in R^n$, $\lim_{t\to0} u^{(\infty)}(x,t) = g(x)$ and $\lim_{t\to0} u_j^{(\infty)}(x,t) = g_j(x)$ for all $j = 1,2,\dots$ by the argument of Lemma 2.2, letting $t \to 0$ in (2.11) we get by Fatou's Lemma,

$$
\int_{R^n} |g - g_j|(x)dx \le \int_{R^n} |f - f_j|(x)dx \to 0 \text{ as } j \to \infty.
$$

Hence $g_j \to g$ in $L^1(R^n)$ as $j \to \infty$. By passing to a subsequence if necessary we may assume without loss of generality that $g_j(x) \to g(x)$ a.e. $x \in R^n$. We next observe that since $\widetilde{g}_j$ satisfies (1.4),

$$\begin{aligned}\widetilde{g}_j(x) &\le C(\|f_j\|_\infty + \|f_j\|_{L^1})\\ &\le C(\|f\|_\infty + \|f\|_{L^1} + 2) \quad \forall x \in R^n, n \ge 3, j = 1, 2, \dots,\end{aligned}$$

for some constants $C > 0$. Hence $\{\widetilde{g}_j\}_{j=1}^\infty$ is uniformly bounded in R^n for $n \ge 3$ and there exist a function $\widetilde{g} \in L^\infty(R^n)$ and a subsequence $\{\widetilde{g}_{j_k}\}_{k=1}^\infty$ of $\{\widetilde{g}_j\}_{j=1}^\infty$ such that $\widetilde{g}_{j_k} \to \widetilde{g}$ weakly in $(L^\infty(K))^*$ for any compact subset K of R^n, $n \ge 3$. Without loss of generality we may assume that $\widetilde{g}_j \to \widetilde{g}$ weakly in $(L^\infty(K))^*$ for any compact subset K of R^n and a.e. $x \in R^n$ as $j \to \infty$.

For $n = 1, 2$, since

$$\begin{aligned}|\nabla \widetilde{g}_j| &\le C(1 + \|f_j\|_{L^\infty}^{p-1})(\|f_j\|_{L^\infty} + \|f_j\|_{L^1}) \quad \forall j = 1, 2, \dots,\\ &\le C(2 + \|f\|_{L^\infty}^{p-1})(\|f\|_{L^\infty} + \|f\|_{L^1} + 2) \quad \forall j = 1, 2, \dots\end{aligned} \tag{2.12}$$

by (1.4), $\{\widetilde{g}_j\}_{j=1}^\infty$ is uniformly Lipschitz continuous. Hence by the Ascoli Theorem either $\{\widetilde{g}_j\}_{j=1}^\infty$ has a subsequence (which we may assume to be the sequence itself) that converges uniformly on compact subsets of R^n to a continuous function, or $\widetilde{g}_j(x) \to \infty$ for all $x \in R^n$ as $j \to \infty$ for $n = 1, 2$. We claim that the latter case is not possible. To prove the claim we let $\widetilde{g} = \lim_{j\to\infty} \widetilde{g}_j$ and let

$$\begin{aligned}&E = \{x \in R^n : g_j(x) \to g(x) \text{ and } \widetilde{g}_j \to \widetilde{g}(x) \text{ as } j \to \infty\},\\ &E_1 = \bigcup_{j=1}^\infty \{x \in R^n : g_j(x) < 1 \text{ and } \widetilde{g}_j \ne 0\},\\ &E_0 = E \cap \{g < 1\} \setminus E_1.\end{aligned}$$

For any $x_0 \in E_0$, since $g_j(x_0) \to g(x_0) < 1$ as $j \to \infty$, there exists $j_0 \in Z^+$ such that $g_j(x_0) < 1$ $\forall j \ge j_0$. Since $x_0 \notin E_1$, $\widetilde{g}_j(x_0) = 0$ for all $j \ge j_0$. Letting $j \to \infty$, we have $\widetilde{g}(x_0) = 0$. Since $|\{g < 1\} \setminus E_0| = 0$ and $|E_1| = 0$ by Lemma 2.3, $\widetilde{g}(x_0) = 0$ a.e. $x_0 \in \{g < 1\}$. Since $\int g < \infty$, $|\{g < 1\}| > 0$. Hence there exists x_0 such that $\widetilde{g}(x_0) = 0$. Thus the claim follows. Hence $\widetilde{g}_j$ converges uniformly to $\widetilde{g}$ on every compact subset of R^n as $j \to \infty$ for $n = 1, 2$. Thus $\widetilde{g} \in L^\infty_{loc}(R^n)$, $\widetilde{g} \ge 0$, and (0.4) holds. We are now ready to show that g, $\widetilde{g}$ satisfy (0.3) and (0.5).

Since $\{|\nabla \widetilde{g}_j|\}_{j=1}^\infty$ is uniformly bounded in R^n for $n = 1, 2$ by (2.12), we may assume without loss of generality that for $n = 1, 2$,

$$\nabla \widetilde{g}_j \to \nabla \widetilde{g} \quad \text{weakly in } (L^\infty(K))^* \quad \text{and a.e. } x \in R^n \quad \text{as } j \to \infty.$$

Putting $\widetilde{g} = \widetilde{g}_j$ in (0.5), and letting $j \to \infty$, we get by Fatou's Lemma that (0.5) holds for $\widetilde{g}$.

We next observe that by Theorem 2.1, $u_j^{(\infty)}$ satisfies, for all $\eta \in C_0^\infty(R^n)$, $0 < t < 1$,

$$\begin{aligned}
&\int_{R^n} \widetilde{g}_j \Delta\eta dx - \int_0^t \int_{R^n} u_j^{(\infty)p} \eta dx d\tau = \int_{R^n} u_j^{(\infty)}(x,t)\eta(x)dx - \int_{R^n} f_j \eta dx \\
\Rightarrow\quad &\int_{R^n} \widetilde{g}\Delta\eta dx - \int_0^t \int_{R^n} u^{(\infty)p}\eta dx d\tau \\
&\qquad = \int_{R^n} u^{(\infty)}(x,t)\eta(x)dx - \int_{R^n} f\eta dx \text{ as } j \to \infty \\
\Rightarrow\quad &\int_{R^n} \widetilde{g}\Delta\eta dx = \int_{R^n} g(x)\eta(x)dx - \int_{R^n} f\eta dx \qquad \text{as } t \to 0 \\
\Rightarrow\quad &g - \Delta\widetilde{g} = f \qquad\qquad \text{in } \mathcal{D}'(R^n).
\end{aligned}$$

Hence (0.3) holds. By the same argument as the proof of Lemma 2.2, $u^{(\infty)}$ and g satisfy (2.6) and (2.7). The theorem then follows from Lemma 1.5, Theorem 1.6 and the uniqueness of a solution of the ODE (2.6), (2.7). □

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